Finite Volume Methods for the EMI Model of Electrocardiology : From Admissible to General Meshes

3rd MICROCARD Workshop / Strasbourg

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Outline

- Model to be solved
- Finite Volume Schemes
- Convergence Analysis and Error Estimates
- Drawbacks of TPFA
- More Flexible Scheme
- Conclusion and Outlook

Microscopic Bidomain Model

$$
\begin{aligned}
-\nabla.(\sigma_k \nabla u_k) &= 0 && \text{in } \Omega_k,\ k = 0, 1, \\
-\sigma_0 \nabla u_0 \cdot n_0 = \sigma_1 \nabla u_1 \cdot n_1 &= -\left(c_m \partial_t v + I_{\text{ion}}(v)\right) && \text{on } \Sigma := \overline{\Omega}_0 \cap \overline{\Omega}_1, \\
-\sigma_0 \nabla u_0 . n_0 &= g^{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}} \\
u_0 &= g^{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}} \\
v &= u_1 - u_0 && \text{on } \Sigma
\end{aligned}
\qquad (1)
$$

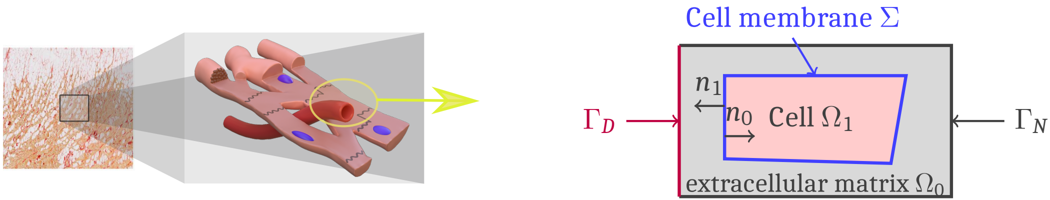
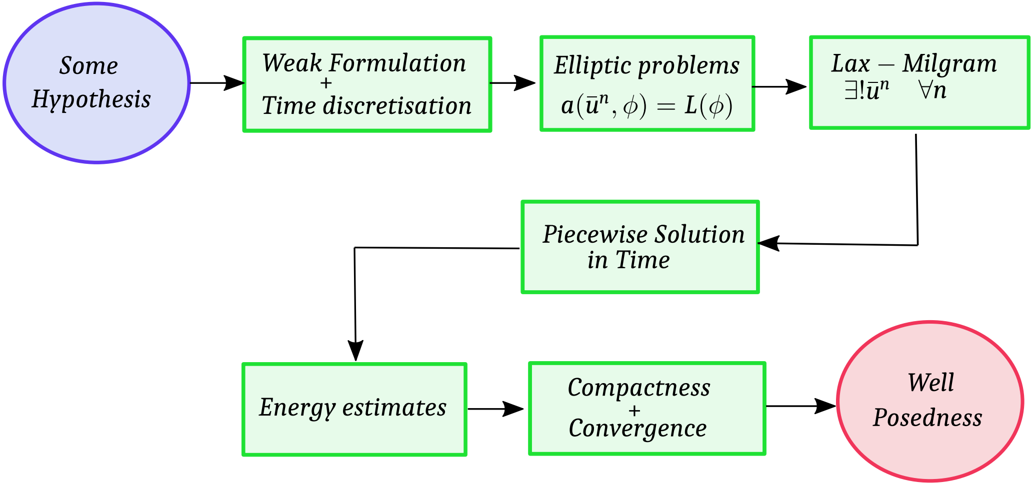


Figure: Geometrical set up of EMI model

Well-Posedness of μ-Model[1] (One cell + Gap-junctions)



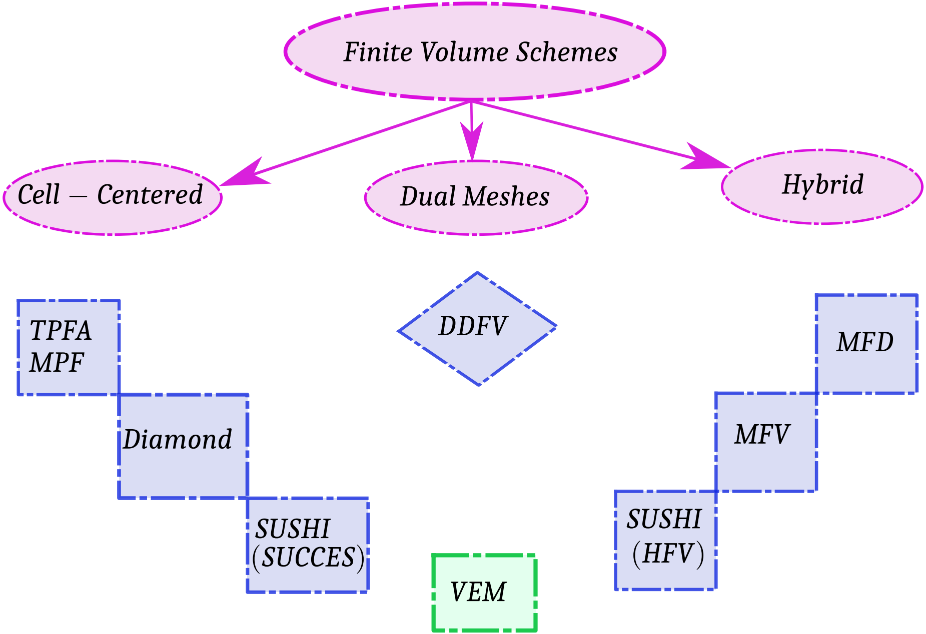
[1] P.E. Bécue. Modélisation et simulation de l'électrophysiologie cardiaque à l'échelle microscopique. UB, 2018.

Finite Volume Methods

Quick Bibliography

- Jerome Droniou, ”Finite Volume Schemes for Diffusion Equations: Introduction to and Review of Modern Methods”, 2014.
- Robert Eymard 1 , Thierry Gallouët 2 and Raphaèle Herbin 3 October 2006. Finite Volume Methods. Handbook of Numerical Analysis, P.G. Ciarlet, J.L. Lions eds, vol 7, pp 713-1020.
- Jérôme Droniou, Robert Eymard, Thierry Gallouët, Cindy Guichard, Raphaele Herbin. The gradient discretisation method. Springer International Publishing AG, 82, 2018.

Review Finite Volume



Cell-Centered Finite Volume

Philosophy of the "Cell-Centered" FV Schemes

- Involve a single unknown u_K per K, taken as approximation of the value of u at each cell.
- Writing the balance equation over each cell, for which we associate the linear system.

They are based on three principle ideas:

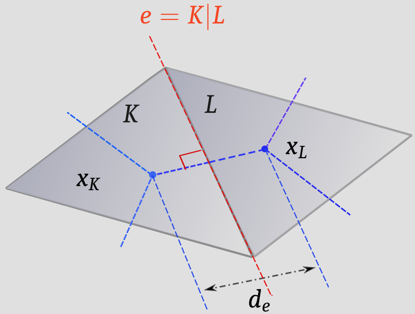
- Subdivision of the spatial domain into subsets called Control Volumes.
- Integration of the equation to be solved over the Control Volumes.
- Approximation of the derivative (flux) appearing after integration.

Admissible [2] Orthogonal Meshes $(\mathcal{T}, \mathcal{E}, \mathcal{P})$

Notations

- $\mathcal{T} = \mathcal{T}_0 \times \mathcal{T}_1 = \{K\}$.
- $\mathcal{E} = \{e := K|L \text{ or } K|.\}$.
- $\mathcal{P} = \{x_K \in K, \forall K \in \mathcal{T}\}$.
- $\text{size}(\mathcal{T}) := \max_{K\in\mathcal{T}} \text{diam}(K)$.
- $\mathcal{E} : \mathcal{E}^1_{\text{int}}, \mathcal{E}^0_{\text{int}}, \mathcal{E}_\Sigma, \mathcal{E}^{\text{D}}_{\text{ext}}, \mathcal{E}^{\text{N}}_{\text{ext}}$.
- $|e|, |K|, n_{K,e}, d_e, d_{K,e}$.
- $T = \Delta t N, t^n = n\Delta t, \forall n \in \{1, ..., N\}$.

Admissibility



[2] R.Eymard, T. Callouet and R. Herbin, Finite volume methods,October 2006. Handbook of Numerical Analysis,vol 7.

FV Scheme Construction

Exact Balance Equations

The integral form of Eq (1.1) on **"any cell** $K \in \mathcal{T}$**"** at time t^n reads

$$-\sum_{e \in \mathcal{E}_K} \int_e \sigma_k \nabla u_k(t^n, \cdot) \cdot n_{Ke} = 0. \tag{2}$$

The integral of Eq (1.2) on **"any interface** $e = K|L \in \mathcal{E}_\Sigma$**"** at time t^n reads

$$-\int_e \sigma_0 \nabla u_0(t^n, \cdot) \cdot n_{Ke} = \int_e \sigma_1 \nabla u_1(t^n, \cdot) \cdot n_{Le} = -\left(c_\mathrm{m} \int_e \partial_t v(t^n, \cdot) + \int_e I_\mathrm{ion}(v(t^n, \cdot)) \right). \tag{3}$$

FV Scheme Construction

Exact Balance Equations

The integral form of Eq (1.1) on **"any cell** $K \in \mathcal{T}$**"** at time t^n reads

$$-\sum_{e \in \mathcal{E}_K} \overline{F}^n_{K,e} = 0. \tag{2}$$

The integral of Eq (1.2) on **"any interface** $e = K|L \in \mathcal{E}_\Sigma$**"** at time t^n reads

$$-\overline{F}^n_{K,e} = \overline{F}^n_{L,e} = -\left(c_\mathrm{m} \int_e \partial_t v(t^n, \cdot) + \int_e I_\mathrm{ion}(v(t^n, \cdot)) \right). \tag{3}$$

TPFA Finite Volume Scheme ($N_{\mathcal{T}} \times N_{\Sigma}$ Eq)

Discrete Balance Equations

We look for $(u^n_{\mathcal{T}}, v^n_{\mathcal{T}}) \in \mathbb{R}^{\mathcal{T}} \times \mathbb{R}^{\Sigma}$, such that $\forall n \in \{1, ..., N\}$

$$- \sum_{e \in \mathcal{E}_K} F^n_{Ke} = 0, \qquad \text{for all } K \in \mathcal{T}, \tag{4}$$

$$- F^n_{Ke} = F^n_{Le} = - \left(\frac{c_{\mathrm{m}}}{\Delta t} (v^n_e - v^{n-1}_e) + I_{\mathrm{ion}}(v^{n-1}_e) \right) |e|, \qquad \text{for all } e = K|L \in \mathcal{E}_{\Sigma}, \tag{5}$$

Where the numerical flux F^n_{Ke} reads

$$F^n_{Ke} = \tau_e(u^n_L - u^n_K) \qquad \text{if } e = K|L \in \mathcal{E}_K \cap \mathcal{E}^{\star},\ K \in \mathcal{T}, \tag{6}$$

$$F^n_{Ke} = \tau_e(g^{D,n}_e - u^n_K) \qquad \text{if } e = K| \in \mathcal{E}_K \cap \mathcal{E}^{\mathrm{D}}_{\mathrm{ext}},\ K \in \mathcal{T}_0, \tag{7}$$

$$F^n_{Ke} = - g^{N,n}_e |e| \qquad \text{if } e = K| \in \mathcal{E}_K \cap \mathcal{E}^{\mathrm{N}}_{\mathrm{ext}},\ K \in \mathcal{T}_0, \tag{8}$$

$$F^n_{Ke} = \tau_e(u^n_L - u^n_K - v^n_e) \qquad \text{if } e = K|L \in \mathcal{E}_K \cap \mathcal{E}_{\Sigma},\ K \in \mathcal{T}_0,\ L \in \mathcal{T}_1 \tag{9}$$

The discrete balance equations (8)-(13) have a unique discrete solution $(u^n_{\mathcal{T}}, v^n_{\mathcal{T}})$. (By R.E. & all)

TPFA Finite Volume Scheme ($N_{\mathcal{T}} \times N_{\Sigma}$ Eq)

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- Introduce the "auxiliary" unknowns u_{Ke}, u_{Le} s.t. : $F^n_{Ke} = \tau_{Ke}(u_{K,e} - u^n_K)$, $F^n_{Le} = \tau_{Le}(u_{L,e} - u^n_L)$.
- Thanks to $F^n_{Ke} + F^n_{Le} = 0$, one deduce $u_{K,e} = \frac{\tau_K}{\tau_K + \tau_L} u^n_K + \frac{\tau_L}{\tau_K + \tau_L}(u^n_L - v^n_e)$, $u_{L,e} = \frac{\tau_K}{\tau_K + \tau_L}(u^n_K + v^n_e) + \frac{\tau_L}{\tau_L + \tau_L} u^n_L$

Matrix Form

$$\begin{pmatrix} A & B \\ B^T & C + \frac{c_\mathrm{m}}{\Delta t}|e| \, \mathrm{Id}_{N_\Sigma} \end{pmatrix} \begin{pmatrix} u^n_\mathcal{T} \\ v^n_\mathcal{T} \end{pmatrix} = \begin{pmatrix} G^n \\ -\left(-\frac{c_\mathrm{m}}{\Delta t} v^{n-1}_\mathcal{T} + I_\mathrm{ion}(v^{n-1}_\mathcal{T})\right)|e| \end{pmatrix} \tag{10}$$

- A: square matrix $(N_\mathcal{T} \times N_\mathcal{T})$, with $a_{KL} = -\tau_e$ and $a_{KK} = \sum_{e \in \mathcal{E}_K} \tau_e$.
- B: matrix $(N_\mathcal{T} \times N_\Sigma)$, with $b_{Ke} = \tau_e$ and $b_{Le} = -\tau_e$.
- C: matrix $(N_\Sigma \times N_\Sigma)$, with $c_{ee} = \tau_e$.
- Schur complement: matrix $(N_\Sigma \times N_\Sigma)$

$$(\underbrace{C - B^\intercal A^{-1} B}_{SC}) v^n_\mathcal{T} = -|e| \frac{c_\mathrm{m}}{\Delta t}(v^n_\mathcal{T} - v^{n-1}_\mathcal{T}) + |e| I_\mathrm{ion}(v^{n-1}_\mathcal{T}) - B^\intercal A^{-1} G^n.$$

Linear system (10) is SDP and coercive with a standard discrete H^1 semi-norm and L^2 norm.

Convergence of TPFA Scheme

Convergence Analysis of TPFA Scheme

Outline of the Proof ($\mathbb{C}^2$ regularity)

- Defines errors associated with $(u^n_{\mathcal{T}}, v^n_{\mathcal{T}})$.

$$\eta^n_{\mathcal{T}} = \overline{u}^n_{\mathcal{T}} - u^n_{\mathcal{T}}, \qquad \epsilon^n_{\mathcal{T}} = \overline{v}^n_{\mathcal{T}} - v^n_{\mathcal{T}}.$$

Convergence Analysis of TPFA Scheme

Outline of the Proof ($\mathbb{C}^2$ regularity)

- Defines errors associated with $(u^n_{\mathcal{T}}, v^n_{\mathcal{T}})$.
- Compare $\hat{F}^n_{K,e}$ with $\overline{F}^n_{K,e}$.

$$|e| R^n_{K,e}(u_0, u_1) = \hat{F}^n_{K,e} - \overline{F}^n_{K,e}.$$

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- Defines errors associated with $(u^n_{\mathcal{T}}, v^n_{\mathcal{T}})$.
- Compare $\hat{F}^n_{K,e}$ with $\overline{F}^n_{K,e}$.
- Subtract the **discrete flux balance** equations from **exact flux balance**.

$$-\sum_{e\in\mathcal{E}_K}(\hat{F}^n_{Ke} - F^n_{Ke}) = -\sum_{e\in\mathcal{E}_K}|e|R^n_{Ke},$$

$$-(\hat{F}^n_{Ke} - F^n_{Ke}) = -|e|R^n_{Ke} - (c_{\mathrm{m}}T^n_e + S^n_e)\,|e| - \left(\frac{c_{\mathrm{m}}}{\Delta t}(\eta^n_e - \eta^{n-1}_e) + I_{\mathrm{ion}}(\overline{v}^{n-1}_e) - I_{\mathrm{ion}}(v^{n-1}_e)\right)|e|.$$

Outline of the Proof ($\mathbb{C}^2$ regularity)

- Defines errors associated with $(u_{\mathcal{T}}^n, v_{\mathcal{T}}^n)$.
- Compare $\hat{F}_{K,e}^n$ with $\overline{F}_{K,e}^n$.
- Subtract the **discrete flux balance** equations from **exact flux balance**.
- Error estimate.

$$m' \Delta t |(\epsilon_{\mathcal{T}}^n, \eta_{\mathcal{T}}^n)|_{1,h}^2 + c_{\mathrm{m}} \|\eta_{\mathcal{T}}^n\|_{0,\Sigma}^2 \leq c_{\mathrm{m}} \left(1 + \frac{\lambda}{c_{\mathrm{m}}} \Delta t\right) \|\eta_{\mathcal{T}}^{n-1}\|_{0,\Sigma} \|\eta_{\mathcal{T}}^n\|_{0,\Sigma} + \Delta t R^n \left|(\epsilon_{\mathcal{T}}^n, \eta_{\mathcal{T}}^n)\right|_{1,h} + \Delta t \left(c_{\mathrm{m}} \|T_{\mathcal{T}}^n\|_{0,\Sigma} + \|S_{\mathcal{T}}^n\|_{0,\Sigma}\right) \|\eta_{\mathcal{T}}^n\|_{0,\Sigma},$$

Convergence Analysis of TPFA Scheme

Outline of the Proof ($\mathbb{C}^2$ regularity)

- Defines errors associated with $(u^n_{\mathcal{T}}, v^n_{\mathcal{T}})$.
- Compare $\hat{F}^n_{K,e}$ with $\overline{F}^n_{K,e}$.
- Subtract the **discrete flux balance** equations from **exact flux balance**.
- Error estimate.
- Extracting two inequalities.

$$\|\eta^n_{\mathcal{T}}\|_{0,\Sigma} \leq \underbrace{\exp\left(\frac{3}{2}\frac{\lambda}{c_{\mathrm{m}}}T\right)\left(\|\eta^0_{\mathcal{T}}\|^2_{0,\Sigma} + \frac{1}{m' c_{\mathrm{m}}}R^2_{\mathcal{T}} + \frac{2c_{\mathrm{m}}}{\lambda}T^2_{\mathcal{T}} + \frac{2}{\lambda c_{\mathrm{m}}}S^2_{\mathcal{T}}\right)^{1/2}}_{\theta(\mathrm{size}(\mathcal{T})+\Delta t)} \tag{11}$$

$$\Delta t \sum_{n=1}^{N} |(\eta^n_{\mathcal{T}}, \epsilon^n_{\mathcal{T}})|^2_{1,h} \leq \underbrace{\frac{c_{\mathrm{m}}}{m'}\|\eta^0_{\mathcal{T}}\|^2_{0,\Sigma} + 2c^2_{\mathrm{m}}T^2_{\mathcal{T}} + 2S^2_{\mathcal{T}} + 2(\lambda+1)T \max_{n=1\ldots N}\|\eta^n_{\mathcal{T}}\|^2_{0,\Sigma}.}_{\theta(\mathrm{size}(\mathcal{T})+\Delta t)} \tag{12}$$

Convergence Analysis of TPFA Scheme

Outline of the Proof (H^2 regularity)

- Convex-hull for any K,
  $\mathcal{V}_{K,e} = \{sx + (1-s)x_K, x \in e, s \in [0,1]\}$.
- We assume without loss of generality the following geometry $\Rightarrow$ Figure.
- Regularity conditions on the mesh + Technical computations.
- Consistency error
  $|R^n_{K,e}| \leq C \frac{size\mathcal{T}}{(m(e)d_e)^{1/2}} ||u||_{H^2(\mathcal{V}_{K,e})}$.
- Same generic form of the error estimates like above.

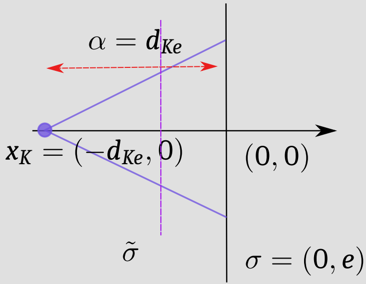

Figure: Convex-Hull

Drawbacks of TPFA

Drawbacks of TPFA

How to find an admissible mesh

- Cartesian meshes: **OK** $\checkmark$.
- Non Cartesian quadrangle meshes: **IMPOSSIBLE**.
- Conforming triangular meshes: **OK** $\checkmark$, **BUT** (maybe $d_e = 0$).
- Non Conforming triangular meshes : **IMPOSSIBLE**.

The discrete gradient is not good

- One direction of gradient approximation.
- Weak convergence of the gradient.

Drawbacks of TPFA

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- One direction of gradient approximation.
- Weak convergence of the gradient.

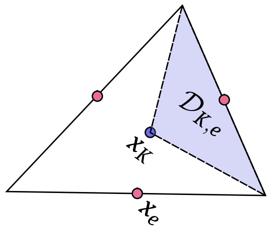
$\Rightarrow$ **Approximation of the gradient using more than 2 points !!**

Flexible scheme: SUSHI

SUSHI[3] (Scheme Using Stabilization and Hybrid Interfaces)

Toy Problem

$$
\begin{aligned}
-\mathrm{div}(\lambda(x)\nabla u) &= f, \text{ on } \Omega \\
u &= 0, \text{ on } \partial\Omega
\end{aligned}
\tag{13}
$$



[3] Discretisation of heterogeneous and anisotropic diffusion prb on general nonconforming meshes SUSHI R. E. & all.

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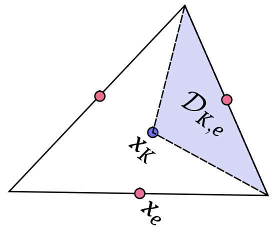

Philosophy of SUSHI Scheme

- Weak formulation with discrete space $\chi_{\mathcal{D},0} = \{(u_K, u_e) \in \mathbb{R}^{\mathcal{T}} \times \mathbb{R}^{\mathcal{E}}, u_e = 0, \forall e \in \mathcal{E}_{ext}\}$.
- Unknowns on cells (SUCCES) and on interfaces (HFV).
- Set of edges splits on barycentric and hybrid edges $\mathcal{E} = \mathcal{E}_B \cup \mathcal{E}_H$; $u_e = \sum_K \gamma_K^e u_K, \forall e \in \mathcal{E}_B$.
- Geometric formula $\Rightarrow \nabla_K u = \frac{1}{|K|} \sum_{e \in \mathcal{E}_K} |e|(u_e - u_K) n_{K,e}$.

$\Rightarrow$ **adapted to μ-Model ?!**

[3] Discretisation of heterogeneous and anisotropic diffusion prb on general nonconforming meshes SUSHI R. E. & all.

Conclusion, Ongoing & Outlook

- **Conclusion**
  - Well-posedness of μ-Model.
  - TPFA Scheme on Admissible mesh
  - Convergence Analysis via Error Estimates with different regularities.
  - Extension to more flexible scheme with general meshes ”SUSHI”.

Conclusion, Ongoing & Outlook

- **Conclusion**
  - Well-posedness of μ-Model.
  - TPFA Scheme on Admissible mesh
  - Convergence Analysis via Error Estimates with different regularities.
  - Extension to more flexible scheme with general meshes ”SUSHI”.
- **Ongoing**
  - Coding TPFA scheme (2D with triangular mesh, and 3D with FV5-point scheme with rectangular mesh).
  - Working on the SUSHI scheme.
  - Time integration IMEX or Rush-Larsen.
- **Outlook**
  - Diamond Scheme.
  - Implementation on OpenCarp?

Contributions

- **Conference Paper (FVCA10 / Strasbourg):**
  — Zeina CHEHADE & Yves Coudière, ”Two-Point Finite Volume Scheme for a Microscopic Bidomain Model of Electrocardiology”.

- **Article:**
  — P.E. Bécue, Zeina CHEHADE, Yves Coudière, ”Existence of a Solution for a Micrsocopic Bidomain Model of Electrocardiology”. (Reduction)

Thank You For Your Attention!

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